Written test Statistics for Electronics (Tuesday 15.15 group)

Task 1: Given a four element probability space $\Omega = \{(a,a), (a,b), (b,a), (b,b)\}$ with probabilities $P((a,a)) = P((a,b)) = P((b,a)) = P((b,b)) = \frac{1}{4}$.

a) Define random variables X and Y on Ω such that X and Y are independent

b) compute the expectation of $X \cdot Y$

c) Find two events A and B on Ω such that A and B are disjoint and compute $P(A \cup B)$.

d) Find two events C and D on Ω such that C and D are independent and compute $P(C \cap D)$

Task 2: given a continuos random variable X with values in $[1, \infty)$ and cumulative distribution function $F_X(z) = \left\{ \begin{array}{c} 1 - z^{-2} \text{ for } z \geq 1 \\ 0 \text{ for } z < 1 \end{array} \right|$. Compute the following:

a) The density $\varphi_X(z)$ of the random variable X and sketch the graphs of F and φ

b) The expectation and variance of X

c) The probability $P\{X = 3\}$ that the random variable X has value equal to 3.

d) The probability $P\{0 \leq X \leq 2\}$ that the random variable has value in the interval $[0, 2]$.

Task 3: A fair coin is thrown 40 000 times .

a) Use the central limit theorem to give an approximate upper bound on the probability that head appears at least 20300 times. List of values for the cumulative distribution function $\Phi(z)$ of the standard normal distribution (expectation zero and variance 1):

$\Phi(1) = 0.841\,34; \Phi(1.5) = 0.933\,19; \Phi(2) = 0.977\,25;\ \Phi(2.5) = 0.993\,79; \Phi(3) = 0.998\,65; \Phi(3.5) = 0.999\,77; \Phi(4) = 0.999\,97$

b) (optional): Compare the result obtained in task 6a with an upper bound obtained with the help of Hoffding's inequality. Which bound is better?

Hint : The Hoffding inequality states the following: Let $\{X_i\}_{i=1}^n$ be a sequence of independent identically distributed random variables X_i such that $a \leq X_i \leq b$. Let μ be the expectation of X_i and let $c = b - a$. Then the following holds for all $n \geq 1$ and $t \geq 0$

$$P\left\{\sum_{i=1}^{n} X_i - n\mu \geq t\right\} \leq e^{-\frac{2t^2}{nc^2}} \tag{1}$$

Task 4: Given two independent random variables X and Y with $EX = 1$ and $VarX = 2$ and $EY = 2$ and $VarY = 1$. Use the Chebychev inequality to estimate $\Pr\{|2Y - 3X - 1| \geq 8\}$ from above.

Task 5: Given n independent samples $\{x_i\}_{i=1}^n$ of a random variable X . It is known that the random variable X has a density of the form $\varphi(x) = (a-1)\,x^{-a}$ for $x \geq 1$ and zero otherwise . The parameter $a > 2$ is unknown . Derive a formula for the maximum likelihood estimator $\tilde{a}_{ML}$ of the parameter a, given $\{x_i\}_{i=1}^n$.